NAS1.60:1738

NASA Technical Paper 1738

ORIGINAL
COMPLETED

An Analytical Technique for Approximating Unsteady Aerodynamics in the Time Domain

H. J. Dunn

NOVEMBER 1980

NASA

NASA Technical Paper 1738

An Analytical Technique for Approximating Unsteady Aerodynamics in the Time Domain

H. J. Dunn
Langley Research Center
Hampton, Virginia

National Aeronautics
and Space Administration

**Scientific and Technical
Information Branch**

1980

SUMMARY

An analytical technique is presented for approximating unsteady aerodynamic forces in the time domain. The order of elements of a matrix Padé approximation is postulated, and the resulting polynomial coefficients are determined through a combination of least-squares estimates for the numerator coefficients and a constrained gradient search for denominator coefficients which insures stable approximating functions. The number of differential equations required to represent the aerodynamic forces to a given accuracy tends to be smaller than that employed in certain existing techniques where the denominator coefficients are chosen a priori. The resulting Padé approximation allows the aircraft equations of motion to be formulated in terms of linear ordinary differential equations, a form amenable to the application of optimal control theory methodology. The technique is applied to an aeroelastic, cantilevered, semispan wing whose motion is expressed in terms of five elastic modes. A good estimate is obtained to the generalized unsteady aerodynamic forces on the imaginary axis for elements of the matrix Padé approximation having fourth-order numerator and second-order denominator polynomials.

INTRODUCTION

In the past two decades, control-system designers have seen a shift in the mathematical techniques employed for the analysis and synthesis of dynamic systems. The emphasis has shifted from the so-called "classical" or frequency domain methods developed prior to 1960 by investigators such as Nyquist and Bode, which have played an important role with respect to single input/single output systems, to the more "modern" state-space approach. This shift in emphasis can be attributed to the control designer not only requiring control but also needing or desiring to "optimize" the performance of the control system. A difficulty in using modern control theory for the design of systems to control aeroelastic behavior is the requirement of transforming the unsteady aerodynamic forces, normally provided in the frequency domain, into the time domain. Furthermore, it is highly desirable to use a transformation that results in a dynamic system that is of "reduced order," since the cost of the design increases rapidly with increases in the number of differential equations used to model the system. Several methods that have been presented in the literature to perform this transformation are outlined.

A "Power Series Expansion Method" is presented by Weiss, Tseng, and Morino in reference 1. This method assumes that the unsteady aerodynamic forces can be represented by a power-series expansion in frequency. An advantage of this method is that if the unsteady aerodynamic forces can be represented with two or less terms, then no additional state equations are added to the set of equations used to represent the motion of the aircraft. However, as more terms are added in the expansion, higher order derivatives of the aircraft state variables are introduced. Since unsteady aerodynamic forces are characteristically proportional to the delayed aircraft state, the use of these higher order terms to

approximate the unsteady aerodynamic forces generally will not be as satisfactory as a more appropriate approximation function for the same number of added state equations.

Edwards has proposed, in reference 2, the use of a "Rational Model" to approximate the unsteady aerodynamic forces which add no additional state equations to the mathematical model. This method has been applied only to simple examples because of the lack of a production computer code for the generation of the aerodynamic forces in the Laplace domain.

The method most commonly employed is referred to as the "Least-Squares Method." This method has been used for a number of different aerodynamic configurations in references 3, 4, and 5. The method derives its name from the method used to solve a set of simultaneous equations for the coefficients of an assumed aerodynamic model. This assumed aerodynamic model consists of a rational polynomial in the frequency domain. In order to make the problem linear, the coefficients of the denominator polynomial are assumed known. A least-squares estimator is then employed to solve for the coefficients of the numerator polynomial. These coefficients minimize the mean-square error between the predicted aerodynamic forces and the aerodynamic force data being fit at a fixed set of frequencies. A disadvantage of this method, as it is applied, is the requirement of the user to have a priori knowledge of the denominator polynomial in the assumed aerodynamic model. These parameters, which are usually arbitrarily chosen to be within the dynamic range of the natural frequencies of the aircraft, determine the best accuracy that can be achieved with a given model order. The order of the aerodynamic model is directly proportional to the number of first-order constant coefficient differential equations generated by the algorithm. For the control-systems designer to increase the accuracy of the approximation, he must either spend a considerable amount of time and effort adjusting the parameters of the denominator polynomial or increase the system order and pay the cost of additional state equations in the design cycle.

Another method that has received attention in the literature is the "Matrix Padé Method" as described in references 2, 6, and 7. This method has been applied to a number of simple problems in references 2 and 7. However, when this method is applied to problems of increasing difficulty where there are more than two modes and/or when the predicted aerodynamics are not precise, the resulting approximation function can be unstable. This characteristic of the "Matrix Padé Method" was observed in reference 6. In the present paper, a function is considered stable if a bounded input (i.e., wing motion) always produces a bounded output (i.e., distributed load on the airfoil). This must not be confused with a stable aerodynamic approximation function which, when coupled with the dynamics of the structure, results in a closed-loop system that has a closed-loop instability (i.e., flutter).

The method described in the present paper is an extension of the "Matrix Padé Method" and the "Least-Squares Method." A constrained gradient method is used to find the denominator coefficients of the approximation while constraining the calculated approximation to be stable, and a linear least-squares method is used to solve for the numerator coefficients. By using the method described herein, the control designer need only adjust the order of the approximation

until a satisfactory compromise is reached between the good accuracy obtained when a large number of states are used and the high cost of performing modern control-theory design on large systems.

A brief development of the equations of motion for an elastic airplane are presented. The technique of approximating the unsteady aerodynamic forces is then described. This technique is applied to an aeroelastic, cantilevered, semispan-wing flutter model. Results obtained are compared with those found using the "Least-Squares Method" in terms of both the fit of the oscillatory aerodynamic forces and the predicted behavior of critical system characteristic roots in the vicinity of flutter. The method developed in the present paper utilizes a lower order approximation for the aerodynamic forces than that employed in the "Least-Squares Method" of reference 3. The present method tends in general to require a lower order of approximation than the "Least-Squares Method" because of the inclusion of the denominator polynomials as additional variables in the fit of the oscillatory aerdynamic forces. This results in a smaller number of differential equations for the design model.

SYMBOLS

A	matrix of coefficients used in solution of aerodynamic approximation (see appendix)
$[A_i]$	coefficient matrices of aerodynamic approximation of equation (21)
A_{ij}	amplitude of generalized aerodynamic force $Q_{ij}(t)$
$A\left(M, \frac{\rho V^2}{2}\right)$	matrix of coefficients of first-order differential equations of complete system
a_i	coefficients of numerator polynomial of a Padé approximation
$[B]$	matrix of coefficients (see matrix $[C]$)
B_j	vector of constant coefficients used in solution of aerodynamic approximation (see appendix)
b_ℓ	coefficients of denominator polynomial of a Padé approximation
$[C]$	matrix of coefficients for system equations such that $[C]\ X(t) = [B]\ X(t)$
c_r	wing root chord (see fig. 2)
$c/2$	reference length
$[D]$	generalized damping matrix
$[D_m]$	coefficient matrices of aerodynamic approximation of equation (21)

e	natural base
$F[x(t)], F^{-1}[x(j\omega)]$	Fourier transform pair defined by equation (9)
$[I]$	identity matrix
J	measure of the error between data and approximation
J_j	function minimized by equation (20)
$\bar{J}_j$	measure of the error between data and approximation of jth vibration mode
j	$= \sqrt{-1}$
K	order of numerator polynomial of a Padé approximation
$[K]$	generalized stiffness matrix
k	reduced frequency, $\omega c/2V$
k_ℓ	specific reduced frequency
M	free-stream Mach number
$[M]$	generalized mass matrix
M_i	generalized mass for ith vibration mode
$m(x,y)$	mass per unit area at point x,y
N	order of unsteady aerodynamic Padé approximation and order of denominator polynomial in Padé approximation
N_i	order of unsteady aerodynamic Padé approximation for ith vibration mode
N_k	number of reduced frequencies used in cost function
n	number of vibration modes
$[P_m]$	matrix of coefficients of $[P(j\omega)]$ (see eq. (17))
$[P_{m,j}]$	jth column vector of $[P_m]$
$P_{ij}(jk_\ell)$	ith,jth element of $[P]$ evaluated at frequency k_ℓ
$[P(j\omega)], [P]$	matrix of numerator polynomials of unsteady aerodynamic Padé approximation
$\Delta P(x,y,t)$	time varying total pressure distribution

$\Delta P_j(x,y,t)$ time varying pressure distribution due to impulse of jth mode

$[Q(t)]$ matrix of functions such that ith,jth element is unit impulse response of ith generalized force due to an impulse of jth mode

$Q_i(t)$ generalized aerodynamic force in time domain for ith mode

$Q_{ij}(t)$ ith,jth element of $[Q(t)]$

$[Q(j\omega)]$ Fourier transform of $[Q(t)]$

$[\hat{Q}(j\omega)]$ unsteady aerodynamic Padé approximation

$Q_{ij}(jk_\ell)$ ith,jth element of $[Q(j\omega)]$ evaluated at frequency k_ℓ

$[\hat{Q}_{ij}(jk_\ell)]$ ith,jth element of $[\hat{Q}(j\omega)]$ evaluated at frequency k_ℓ

$q(t)$ vector of generalized coordinate in time domain

$q_i(t)$ ith generalized coordinate in time domain

$q(j\omega)$ Fourier transform of $q(t)$

$[R_\ell]$ matrix of coefficients of $R(j\omega)$ (see eq. (16))

$[\bar{R}(j\omega)]$ matrix $[R(j\omega)]$ with leading coefficient equal to $[I]$

$R_j(jk_\ell)$ jth element of $[R(j\omega)]$ evaluated at frequency k_ℓ

$[R(j\omega)],[R]$ matrix of denominator polynomials of unsteady aerodynamic forces Padé approximation

$r_{\ell,j}$ jth diagonal element of $[R_\ell]$

s semispan length of wing in example

$T(jk)$ general Padé approximation

t time

V free-stream velocity

X_j solution vector used in calculation of aerodynamic approximation (see appendix)

$X(t)$ system state vector

$x(t)_i$ state variable vector

(x,y) scalar product

$z(x,y,t)$ vertical displacement of body at point x,y

ζ_i viscous damping coefficient for ith mode

ρ reference density of fluid

ϕ_i nondimensional mode shape used to define generalized coordinate $q_i(t)$

ϕ_{ij} phase angle of $Q_{ij}(t)$

ω frequency of oscillation

ω_{n_i} natural frequency of vibration of ith mode

$|| \; ||$ norm, $\sqrt{(\; , \;)}$

Dots over symbols denote derivatives with respect to time.

EQUATIONS OF MOTION FOR AEROELASTIC MODEL

The equations of motion for an elastic airplane in straight and level flight can be formulated by using Lagrange's equations of motion with orthogonal modes. By the method of separating variables, the motion on the wing is assumed to be the product of a position function and a time function. It is also assumed that the product of these functions can be represented with sufficient accuracy by a finite series so that the vertical displacement at the point x,y becomes

$$z(x,y,t) = \sum_{i=1}^{n} q_i(t) \; \phi_i(x,y) \tag{1}$$

where $\phi_i(x,y)$ are the nondimensional mode shapes used to represent the system, n is the number of modes included, and $q_i(t)$ represents the generalized coordinate with dimension of length, the particular unit depending upon the system of units being used. If all of the structural damping present in the aircraft is assumed to be viscous in nature, then the equations of motion can be written as

$$M_i \ddot{q}_i(t) + 2\zeta_i M_i \omega_{n_i} \dot{q}_i(t) + \omega_{n_i}^2 M_i \, q_i(t) = -\frac{1}{2} \rho V^2 \, Q_i(t) \tag{2}$$

where ω_{n_i} is the natural frequency of vibration of the ith mode, ζ_i is the viscous damping coefficient for the ith mode, and M_i is the generalized mass of the ith mode defined by

$$M_i = \iint_S \left[m(x,y)\ \phi_i^2(x,y) \right] dx\ dy \tag{3}$$

in which $m(x,y)$ is the mass density of the body at the point x,y and $\iint_S [\ ]\ dx\ dy$ is the integral of the function over the area of the body denoted by S. The term on the right-hand side of equation (2) represents the generalized aerodynamic force applied to the ith mode and is defined by

$$-\frac{1}{2}\rho V^2\ Q_i(t) = \frac{1}{2}\rho V^2 \iint_S \left[\frac{\Delta P(x,y,t)}{\frac{1}{2}\rho V^2}\ \phi_i(x,y) \right] dx\ dy \tag{4}$$

where the term $\rho V^2/2$ is the dynamic pressure of the free stream, $\Delta P(x,y,t)$ is the time varying perturbation in the lifting pressure distribution at the point x,y, and $Q_i(t)$ is the generalized aerodynamic force normalized by the dynamic pressure. The lifting pressure distribution will be assumed to be representable by the following equation:

$$\Delta P(x,y,t) = \Delta P(x,y,0) + \sum_{j=1}^{n} \int_0^t \Delta P_j(x,y,t-\tau)\ q_j(\tau)\ d\tau \tag{5}$$

where $\Delta P_j(x,y,t)$ is the time history of the pressure at the point x,y due to the jth mode being displaced by the unit impulse function at $t = 0$. Substituting this expression for $\Delta P(x,y,t)$ into equation (4) with the initial perturbation in the pressure distribution equal to zero and factoring out the dynamic pressure, $Q_i(t)$ can be expressed as

$$Q_i(t) = \sum_{j=1}^{n} \int_0^t Q_{ij}(t-\tau)\ q_j(\tau)\ d\tau \tag{6}$$

where $Q_{ij}(t)$ is defined by

$$Q_{ij}(t) = -\iint_S \left[\frac{\Delta P_j(x,y,t)}{\frac{1}{2}\rho V^2} \phi_i(x,y) \right] dx\, dy \tag{7}$$

Equation (2) can be rewritten in matrix form as

$$[M]\,\ddot{q}(t) + [D]\,\dot{q}(t) + [K]\,q(t) + \frac{1}{2}\rho V^2 \int_0^t [Q(t-\tau)]\,q(\tau)\,d\tau = \underline{0} \tag{8}$$

where $[M]$ is a diagonal matrix with the ith diagonal element being defined by equation (3), $[D]$ is a diagonal matrix with the ith diagonal element being $2\zeta_i M_i \omega_{n_i}$, $[K]$ is a diagonal matrix with the ith diagonal element being $M_i \omega_{n_i}^2$, $[Q(t)]$ is a matrix with the ith,jth element being defined by equation (7), $q(t)$ is a vector with the ith element being the variable $q_i(t)$, and $\underline{0}$ is a null matrix.

Introducing the Fourier transform pair as

$$\left.\begin{aligned} x(j\omega) &= F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt \\ x(t) &= F^{-1}[x(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega) e^{j\omega t}\,d\omega \end{aligned}\right\} \tag{9}$$

and applying the transform to equation (8) yields

$$\left\{(j\omega)^2\,[M] + (j\omega)\,[D] + [K] + \frac{1}{2}\rho V^2\,[Q(j\omega)]\right\} q(j\omega) = \underline{0} \tag{10}$$

where the following Fourier transform properties have been used:

$$
\left.
\begin{aligned}
(j\omega)^2\, q(j\omega) &= F[\ddot{q}(t)] \\
(j\omega)\, q(j\omega) &= F[\dot{q}(t)] \\
q(j\omega) &= F[q(t)] \\
[Q(j\omega)]\, q(j\omega) &= F\left[\int_0^t [Q(t-\tau)]\, q(\tau)\, d\tau\right]
\end{aligned}
\right\} \qquad (11)
$$

Equation (10) represents a variation of the classical flutter equation, and is valid only for stable aircraft motion.

The ith,jth element of the matrix $[Q(j\omega)]$ represents the Fourier transform of the ith generalized unsteady aerodynamic force, normalized by the dynamic pressure, due to a unit impulse of the jth mode. Alternatively, it can be interpreted as the contribution to the ith generalized force due to a steady oscillation in the jth mode at a frequency of ω, which can be expressed in the time domain as

$$Q_{ij}(t) = A_{ij} \sin(\omega t + \phi_{ij}) \qquad (12)$$

where A_{ij} and ϕ_{ij} are the amplitude and phase angle of the ith,jth element of $[Q(t)]$ or $[Q(j\omega)]$.

DESCRIPTION OF TECHNIQUE

In this section an approximation is developed for the unsteady aerodynamic forces in the frequency domain and, by inversion, in the time domain. The approximation gives an accurate representation of the aerodynamic forces for oscillatory motion provided the assumptions of small perturbation, inviscid flow are valid. The primary sources of errors in the approximation are the availability of only a limited frequency band of aerodynamic force data due to oscillatory motion and the limited order of the matrix Padé approximation employed to fit these data. Quantification of the accuracy of the aerodynamic forces for arbitrary motion requires additional analytical and experimental research and is beyond the scope of this paper.

Theoretical three-dimensional oscillatory aerodynamic forces are normally calculated at a specified Mach number with the body oscillating at a number of reduced frequencies. The reduced frequency k is a nondimensional number that represents the number of radians through which the body oscillates per reference length $c/2$ the body travels through the fluid, or

$$k = \frac{1}{2}\,\frac{\omega c}{V} \qquad (13)$$

where ω is the dimensional circular frequency of oscillation and V is the velocity of the body through the fluid. Vepa, in reference 7, and Edwards, in reference 2, suggest that the unsteady aerodynamic forces may be approximated by a rational function of polynomials in the complex frequency domain (i.e., a Padé approximation in the reduced frequency domain). Padé approximations are classified by the order of the polynomials of the numerator and denominator. In the present paper, the notation $[K,N]$ is used to represent a Padé approximation with a numerator polynomial of order K over a denominator polynomial of order N, or

$$T(jk) = \frac{\sum_{i=0}^{K} (jk)^i a_i}{\sum_{\ell=0}^{N} (jk)^\ell b_\ell} \tag{14}$$

where $T(jk)$ is the approximation of the function, a_i are the coefficients of the numerator polynomial, b_ℓ are the coefficients of the denominator (the coefficient b_N is set to unity), and k is the independent variable of the approximation. Vepa, in reference 7, and Edwards, in reference 2, have suggested the use of an $[N+1,N]$ Padé approximation based on the high-frequency asymptotic behavior of unsteady aerodynamics as predicted by piston theory. Since the $[N+1,N]$ approximation is a special case of the $[N+2,N]$ Padé approximation and since it is desired to approximate the unsteady aerodynamic forces only over a specified frequency range (thereby ignoring the high-frequency behavior), an $[N+2,N]$ Padé approximation was employed for the development of the technique described herein. This is also consistent with the order of the approximation used in references 3 to 5 for the "Least-Squares Method."

The objective is to find a set of stable $[N+2,N]$ Padé approximations which best fit the matrix $[Q(j\omega)]$ of equation (10) at a discrete set of frequencies. The parameter N represents the number of terms used to represent the lag in the development of the circulation (for the two-dimensional case they would approximate the Theodorsen circulation function); henceforth, N is referred to as the order of the unsteady aerodynamic Padé approximation. Also, the term "Padé approximation" refers to the unsteady aerodynamic Padé approximation described in the present paper. A necessary and sufficient condition for stability of the Padé approximation is that the roots of the denominator polynomial of the Padé approximation have negative real parts. Furthermore, it is appropriate, from the number of differential equations generated by the algorithm and the amount of computational resources required by the algorithm, to require that the denominator polynomial be the same for every column of the Padé approximation. This forces the approximations of the unsteady aerodynamic forces that are dependent on the motion of a particular mode to be a function

of the variables that approximate the lag in development of the circulation for that mode and independent of the motion of the other modes. This results in a savings in the number of equations used to approximate the system because one set of equations are used to represent the lag in the development of the circulation per mode rather than per element as in the general Padé approximation. The problem is formulated as follows: let the matrix $[\hat{Q}(j\omega)]$ be the approximation in the frequency domain of the unsteady aerodynamic forces; let $[P(j\omega)]$ be a matrix of elements such that the ith,jth element represents the numerator polynomial of the Padé approximation; and let $[R(j\omega)]$ be a diagonal matrix such that the jth diagonal element is the denominator polynomial of the Padé approximation which best fits the jth column of the matrix $[Q(j\omega)]$ at a discrete set of frequencies. Then,

$$[\hat{Q}(j\omega)] = [P(j\omega)]\,[R(j\omega)]^{-1} \tag{15}$$

where

$$[R(j\omega)] = (j\omega)^N \left(\frac{c}{2V}\right)^N [I] + \sum_{\ell=0}^{N-1} (j\omega)^\ell \left(\frac{c}{2V}\right)^\ell [R_\ell] \tag{16}$$

$$[P(j\omega)] = \sum_{m=0}^{N+2} (j\omega)^m \left(\frac{c}{2V}\right)^m [P_m] \tag{17}$$

The details of the method of solution for finding the matrices $[R_\ell]$ and $[P_m]$ of equations (16) and (17) are given in the appendix. The method is outlined here. The problem is to find the matrices $[P_m]$ and $[R_\ell]$ such that $[\hat{Q}(j\omega)]$ is a good approximation to $[Q(j\omega)]$ subject to the constraints that the roots of the polynomials defined by $[R]$ in equation (15) have negative real parts. This could be accomplished by minimizing

$$J = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\ell=1}^{N_k} ||Q_{ij}(jk_\ell) - \hat{Q}_{ij}(jk_\ell)||^2 \tag{18}$$

while satisfying the stability constraints. In equation (18), J is the cost of the approximation, N_k is the number of reduced frequencies k_ℓ at which

aerodynamic force data are available, $Q_{ij}(jk\ell)$ is the ith,jth element of $[Q(jk)]$ evaluated at the frequency $k\ell$, $\hat{Q}_{ij}(jk\ell)$ is the Padé approximation for the ith,jth element of $[Q(jk)]$ evaluated at the frequency $k\ell$, and $||\ ||$ is a metric norm as defined in the appendix. By noting that the influence of $[P]$ and $[P]$ on the cost is column dependent, by virture of $[R]$ being diagonal, the problem can be reduced into n smaller problems by minimizing

$$\bar{J}_j = \frac{1}{2} \sum_{i=1}^{n} \sum_{\ell=1}^{N_k} ||Q_{ij}(jk\ell) - \hat{Q}_{ij}(jk\ell)||^2 \tag{19}$$

the function actually minimized in this paper is

$$J_j = \frac{1}{2} \sum_{i=1}^{n} \sum_{\ell=1}^{N_k} ||Q_{ij}(jk\ell)\ R_j(jk\ell) - P_{ij}(jk\ell)||^2 \tag{20}$$

where $R_j(jk\ell)$ is the jth element of the matrix $[R]$ and $P_{ij}(jk\ell)$ is the ith,jth element of $[P]$ evaluated at the reduced frequency $k\ell$. Minimizing equation (20) rather than equation (19) results in larger errors at the lower frequencies. It is shown subsequently to give good results. A numerical gradient procedure is used to find $[R(j\omega)]$ so that equation (20) is minimized while constraining the roots of the polynomial to have negative real parts. At each gradient calculation, a linear least-squares estimator is used to calculate the elements of $[P(j\omega)]$.

Equation (15) can be rewritten as

$$[\hat{Q}(j\omega)] = [A_0] + (j\omega)\left(\frac{c}{2V}\right)[A_1] + (j\omega)^2\left(\frac{c}{2V}\right)^2 [A_2]$$

$$+ \left\{\sum_{m=0}^{N-1} (j\omega)^m \left(\frac{c}{2V}\right)^{m-N} [D_m]\right\} [\bar{R}(j\omega)]^{-1} \tag{21}$$

where $[\bar{R}(j\omega)]$ is equal to $[R(j\omega)]$ with the leading coefficient set to the identity matrix, or

$$[\bar{R}(j\omega)] = \left(\frac{c}{2V}\right)^{-N} [R(j\omega)] \tag{22}$$

Taking equation (10) and substituting $[\hat{Q}(j\omega)]$ of equation (21) for $[Q(j\omega)]$ results in

$$\left\{\left([M] + \frac{\rho}{8} c^2 [A_2]\right)(j\omega)^2 + \left([D] + \frac{\rho}{4} Vc [A_1]\right)(j\omega) + \left([K] + \frac{1}{2} \rho V^2 [A_0]\right) + \left[\frac{1}{2} \rho V^2 \sum_{m=0}^{N-1} [D_m] \left(\frac{c}{2V}\right)^{m-N} (j\omega)^m\right][\bar{R}(j\omega)]^{-1}\right\} q(j\omega) = \underline{0} \tag{23}$$

The state-space formulation of this approximation to the aeroelastic model is derived by denoting the state variables as follows:

$$\left.\begin{aligned} x(t)_1 &= F^{-1}[q(j\omega)] \\ x(t)_2 &= F^{-1}[(j\omega)\ q(j\omega)] = \dot{x}(t)_1 \\ x(t)_3 &= F^{-1}[\bar{R}^{-1}(j\omega)\ q(j\omega)] \\ x(t)_4 &= F^{-1}[(j\omega)\ \bar{R}^{-1}(j\omega)\ q(j\omega)] \\ &\vdots \\ x(t)_{N+2} &= F^{-1}[(j\omega)^{N-1}\ \bar{R}^{-1}(j\omega)\ q(j\omega)] = \dot{x}(t)_{N+1} \end{aligned}\right\} \tag{24}$$

Using equation (24), equation (23) can be rearranged as

$$\dot{x}(t)_1 = x(t)_2 \tag{25a}$$

$$-\left([M] + \frac{\rho}{8} c^2 [A_2]\right) \dot{x}(t)_2 = \left([K] + \frac{\rho}{2} V^2 [A_0]\right) x(t)_1 + \left([D] + \frac{\rho}{4} Vc [A_1]\right) x(t)_2 + \frac{1}{2} \rho V^2 \sum_{m=0}^{N-1} [D_m] \left(\frac{c}{2V}\right)^{m-N} x(t)_{m+3} \tag{25b}$$

$$\dot{x}(t)_3 = x(t)_4 \tag{25c}$$

$$\vdots$$

$$\dot{x}(t)_{N+2} = x(t)_1 - \sum_{\ell=0}^{N-1} \left(\frac{c}{2V}\right)^{\ell-N} [R_\ell]\, x(t)_{\ell+3} \tag{25d}$$

Equations (25) represent a set of first-order constant-coefficient differential equations which can be put into the form of

$$\dot{X}(t) = A\left(M, \frac{\rho V^2}{2}\right) X(t) = [C]^{-1}\,[B]\,X(t) \tag{26}$$

The matrix $[C]$ for $N = 4$ is

$$[C] = \begin{bmatrix} [I] & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & [M] + \frac{\rho}{8} c^2 [A_2] & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & [I] & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & [I] & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & [I] & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & [I] \end{bmatrix} \tag{27}$$

and the matrix $[B]$ for $N = 4$ is

$$[B] = \begin{bmatrix} \underline{0} & [I] & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ [K] + \frac{1}{2}\rho V^2 [A_0] & [D] + \frac{\rho}{4} Vc\,[A_1] & \frac{1}{2}\rho V^2 [D_0]\left(\frac{2V}{c}\right)^4 & \frac{1}{2}\rho V^2 [D_1]\left(\frac{2V}{c}\right)^3 & \frac{1}{2}\rho V^2 [D_2]\left(\frac{2V}{c}\right)^2 & \frac{1}{2}\rho V^2 [D_3]\left(\frac{2V}{c}\right) \\ \underline{0} & \underline{0} & \underline{0} & [I] & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & [I] & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & [I] \\ [I] & \underline{0} & -[R_0]\left(\frac{2V}{c}\right)^4 & -[R_1]\left(\frac{2V}{c}\right)^3 & -[R_2]\left(\frac{2V}{c}\right)^2 & -[R_3]\left(\frac{2V}{c}\right) \end{bmatrix} \tag{28}$$

The dimensions of the matrix $A\left(M, \frac{\rho V^2}{2}\right)$ are a function of the number of modes used to represent the system n and the order of the Padé approximation used N. For clarity in the above development, the order of the approximation was

made the same for every mode shape in the system. This allowed the equations of motion to be derived with vector notation instead of using the individual scalar equation for each mode. In practice, the order of the approximation for each mode is adjusted until the error between the approximation and the aerodynamic data is below a preset value. The number of first-order differential equations is equal to

$$2n + \sum_{i=1}^{n} N_i \tag{29}$$

where N_i is the order of the Padé approximation used to approximate the unsteady aerodynamic forces for the ith mode.

The eigenvalues of $A\left(M, \frac{\rho V^2}{2}\right)$ include roots resulting from the unsteady-aerodynamic-forces approximation as well as the roots of the classical flutter equation. The value of the dynamic pressure $\rho V^2/2$ that results in an oscillatory eigenvalue with the real part of the eigenvalue equal to zero is the flutter point for the vehicle at the particular Mach number for which the unsteady aerodynamic data were calculated. Determining the flutter point for a range of Mach numbers determines the flutter boundary of the vehicle.

RESULTS AND DISCUSSION

The method developed in this paper is compared with the "Least-Squares Method" of references 3, 4, and 5 by application to the example of reference 5 which is an aeroelastic semispan wind-tunnel model with an assumed plane of symmetry at the root. The wing geometry is given in figure 1, and the generalized masses and frequencies are presented in table I for the first 10 elastic modes. For the model used in this example, modes 1, 2, 4, 5, and 6 were used to represent the wing. Structural damping was assumed to be negligible. The oscillatory aerodynamic forces were calculated using a doublet-lattice technique similar to that described in reference 8. In order to calculate the pressure distribution on an oscillating wing undergoing simple harmonic motion, the lifting surface is subdivided into an array of trapezoidal boxes arranged in strips parallel to the airstream as shown in figure 2. The lifting surface is then represented by a lattice of doublets located at the quarter chord of each box. The downwash condition is satisfied at the three-quarter chord of each box by equating it to the downwash resulting from the slope and deflection rate of each structural mode. The lifting surface was divided into 210 boxes arranged in 30 strips spanwise with seven boxes chordwise. Oscillatory aerodynamic forces were calculated at six reduced frequencies (k = 0, 0.1, 0.3, 0.5, 0.7, and 0.9).

The unsteady aerodynamic forces were approximated through the use of the method described in this present paper to calculate the matrix coefficients of

equation (21) and equation (22). First- and second-order Padé approximations were used for all five modes. The approximations (calculated in square meters) for $\hat{Q}_{11}$, $\hat{Q}_{21}$, $\hat{Q}_{12}$, and $\hat{Q}_{22}$ unsteady aerodynamic forces for $N = 2$ are

$$\left.\begin{aligned}
\hat{Q}_{11} &= (3.39 \times 10^{-8})(j\omega)^2 + (1.18 \times 10^{-4})(j\omega) + (1.62 \times 10^{-2}) \\
&\quad - \frac{(2.03 \times 10^{3})(j\omega)}{j\omega + 5.52 \times 10^{1}} - \frac{(1.59 \times 10^{2})(j\omega)}{j\omega + 7.15 \times 10^{2}} \\
\hat{Q}_{21} &= (1.13 \times 10^{-8})(j\omega)^2 - (2.28 \times 10^{-5})(j\omega) + (2.56 \times 10^{-2}) \\
&\quad - \frac{(5.25 \times 10^{-3})(j\omega)}{j\omega + 5.52 \times 10^{1}} + \frac{(6.85 \times 10^{-2})(j\omega)}{j\omega + 7.15 \times 10^{2}} \\
\hat{Q}_{12} &= (1.15 \times 10^{-7})(j\omega)^2 - (1.51 \times 10^{-4})(j\omega) - (1.69 \times 10^{-1}) \\
&\quad + \frac{(3.42 \times 10^{-2})(j\omega)}{j\omega + 5.80 \times 10^{1}} - \frac{(1.40 \times 10^{-2})(j\omega)}{j\omega + 4.26 \times 10^{2}} \\
\hat{Q}_{22} &= -(8.47 \times 10^{-8})(j\omega)^2 + (3.50 \times 10^{-4})(j\omega) - (2.47 \times 10^{-1}) \\
&\quad + \frac{(6.18 \times 10^{-2})(j\omega)}{j\omega + 5.80 \times 10^{1}} + \frac{(8.64 \times 10^{-2})(j\omega)}{j\omega + 4.26 \times 10^{2}}
\end{aligned}\right\} \quad (30)$$

The least-squares approximation for the same unsteady aerodynamic forces with the same a priori parameters as in reference 5 are

$$\begin{aligned}
\hat{Q}_{11} &= (3.81 \times 10^{-8})(j\omega)^2 + (1.12 \times 10^{-4})(j\omega) + (1.63 \times 10^{-2}) \\
&\quad - \frac{(5.37 \times 10^{-3})(j\omega)}{j\omega + 8.50 \times 10^{1}} + \frac{(8.49 \times 10^{-3})(j\omega)}{j\omega + 1.70 \times 10^{2}} \\
&\quad - \frac{(1.23 \times 10^{-3})(j\omega)}{j\omega + 2.55 \times 10^{2}} - \frac{(1.08 \times 10^{-2})(j\omega)}{j\omega + 3.40 \times 10^{2}}
\end{aligned} \quad (31a)$$

$$\hat{Q}_{21} = -(1.12 \times 10^{-9})(j\omega)^2 + (1.48 \times 10^{-6})(j\omega) + (2.55 \times 10^{-2})$$

$$- \frac{(1.56 \times 10^{-2})(j\omega)}{j\omega + 8.50 \times 10^{1}} + \frac{(5.80 \times 10^{-2})(j\omega)}{j\omega + 1.70 \times 10^{2}}$$

$$- \frac{(1.47 \times 10^{-1})(j\omega)}{j\omega + 2.55 \times 10^{2}} + \frac{(1.37 \times 10^{-1})(j\omega)}{j\omega + 3.40 \times 10^{2}} \qquad (31b)$$

$$\hat{Q}_{12} = -(7.01 \times 10^{-9})(j\omega)^2 - (4.58 \times 10^{-2})(j\omega) - (1.68 \times 10^{-1})$$

$$+ \frac{(7.45 \times 10^{-2})(j\omega)}{j\omega + 8.50 \times 10^{1}} - \frac{(2.61 \times 10^{-1})(j\omega)}{j\omega + 1.70 \times 10^{2}}$$

$$+ \frac{(6.41 \times 10^{-1})(j\omega)}{j\omega + 2.55 \times 10^{2}} - \frac{(5.17 \times 10^{-1})(j\omega)}{j\omega + 3.40 \times 10^{2}} \qquad (31c)$$

$$\hat{Q}_{22} = -(1.50 \times 10^{-7})(j\omega)^2 + (4.32 \times 10^{-4})(j\omega) - (2.46 \times 10^{-1})$$

$$+ \frac{(1.53 \times 10^{-1})(j\omega)}{j\omega + 8.50 \times 10^{1}} - \frac{(3.93 \times 10^{-1})(j\omega)}{j\omega + 1.70 \times 10^{2}}$$

$$+ \frac{(6.88 \times 10^{-1})(j\omega)}{j\omega + 2.55 \times 10^{2}} - \frac{(3.69 \times 10^{-1})(j\omega)}{j\omega + 3.40 \times 10^{2}} \qquad (31d)$$

The fit, to the doublet-lattice data, achieved by the approximations described by equation (30) and equations (31) is presented in figure 3. For this case, figure 3 indicates that both the "Least-Squares Method" with four terms and the Padé approximation method presented in the present paper with two terms are accurate approximations for the Q_{11}, Q_{21}, and Q_{22} aerodynamic forces. The Padé approximation for the Q_{12} aerodynamic force does not do as well as the "Least-Squares Method." But, an important point to be noted is that the Padé method requires 33 percent fewer state equations than employed with the "Least-Squares Method." In addition, the a priori parameters required by the "Least-Squares Method" are not needed by the Padé approximation method. The frequencies and damping ratios predicted by the two methods are described in figure 4. Figure 4 also presents results of a first-order Padé approximation when used in the stability analysis. The figure indicates that the predicted damping

ratio for the first mode is not as accurate for the first-order Padé approximation as for the second-order Padé approximation. Since the change in the frequency and the damping ratio with respect to a change in the dynamic pressure is predicted correctly near the flutter dynamic pressure, it may be possible to use a first-order Padé approximation in the design stage while adjusting the design parameters accordly to account for the error. If this could be done, a 50-percent reduction in the number of state equations needed to represent the system could be realized. As shown in figure 4, the second-order Padé approximation estimates the dynamic pressure at which the system becomes unstable to about the same accuracy as does the "Least-Squares Method."

CONCLUDING REMARKS

An analytical method for approximating time-domain aerodynamic forces has been presented. The method is based on approximating the oscillatory aerodynamic forces with Padé approximations in the reduced frequency domain. An approximate time-domain representation is then developed, assuming stable aircraft motion, through the use of the inverse Fourier transform. The analytical method is applied to an aeroelastic wind-tunnel model and showed good agreement with previously used analytical methods for predicting the flutter point and stability trends. Some of the important results of this work are as follows:

1. All the parameters of the aerodynamic model are calculated without a priori knowledge, unlike the currently formulated "Least-Squares Method."

2. The resulting approximations are stable in the sense that a bounded input will produce a bounded generalized aerodynamic force output.

3. The number of differential equations required to represent the aerodynamic force to a given accuracy tends to be smaller than employed in certain existing techniques where the denominator coefficients are chosen a priori. The observed reduction in the set of system equations was 33 percent from that employed in the "Least-Squares Method."

Langley Research Center
National Aeronautics and Space Administration
Hampton, VA 23665
September 5, 1980

APPENDIX

APPROXIMATION OF UNSTEADY AERODYNAMIC FORCES

The generalized aerodynamic forces can be thought of as a matrix of elements such that the ith,jth elements represent the generalized force that is applied to the ith structural mode due to a pressure distribution caused by the jth mode oscillating at a reduced frequency. In practice, the generalized aerodynamic force matrix is calculated for a number of specific Mach numbers and reduced frequencies. This appendix describes a technique which calculates an approximation to the unsteady aerodynamic forces as a rational polynomial in reduced frequency. Details are only provided for the first-order Padé approximation (N = 1).

Consider equations (15), (16), and (17), with k being substituted for $\omega c/2V$ and N = 1. Then

$$[\hat{Q}(jk)] = [P(jk)][R(jk)]^{-1} \tag{A1}$$

where

$$[R(jk)] = [R_0] + (jk)\,[I] \tag{A2}$$

and

$$[P(jk)] = [P_0] + (jk)\,[P_1] + (jk)^2\,[P_2] + (jk)^3\,[P_3] \tag{A3}$$

The matrices $[R_0]$, $[P_0]$, $[P_1]$, $[P_2]$, and $[P_3]$ must be found such that the following equation is made as small as possible and the roots of the polynomials of equation (A2) have negative real parts

$$J = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\ell=1}^{N_k} ||Q_{ij}(jk\ell) - \hat{Q}_{ij}(jk\ell)||^2 \tag{A4}$$

The function $||\ ||$ is the norm of the argument and is defined as

$$||x|| = \sqrt{(x,x)} \tag{A5}$$

where the function (,) is the scalar product which has the following metric properties:

$(x,x) \geq 0$ (equality holds if and only if $x = 0$)

$(y,x) =$ Conjugate (x,y)

$(x,cy) = c(x,y)$ (c = Constant)

$(x,y_1+y_2) = (x,y_1) + (x,y_2)$

The minimization of equation (A4) can be separated into n smaller problems by noting that the influence of $[P]$ and $[R]$ on the cost is column dependent since $[R]$ is diagonal. Therefore, the problem is to find the jth column vectors $[P_{0,j}]$, $[P_{1,j}]$, $[P_{2,j}]$, and $[P_{3,j}]$ of the matrices $[P_0]$, $[P_1]$, $[P_2]$, and $[P_3]$ and the jth element $r_{0,j}$ of the diagonal matrix $[R_0]$

$$J_j = \frac{1}{2} \sum_{i=1}^{n} \sum_{\ell=1}^{N_k} ||Q_{ij}(jk\ell)\,[r_{0,j} + (jk\ell)] - [P_{0,j}] - (jk\ell)\,[P_{1,j}]$$

$$- (jk\ell)^2\,[P_{2,j}] - (jk\ell)^3\,[P_{3,j}]||^2 \tag{A6}$$

The effect of minimizing equation (A6) instead of equation (A4) is to cause the maximum relative error of the Padé approximation to be at the lower frequencies. For a given $r_{0,j}$, equation (A6) can be rewritten as

$$J_j = \frac{1}{2} \sum_{i=1}^{n} ||B_j - AX_j||^2 \tag{A7}$$

where

APPENDIX

$$B_j = \begin{bmatrix} \mathrm{Re}\left[Q_{ij}(r_{0,j} - jk_1)\right] \\ \vdots \\ \mathrm{Re}\left[Q_{ij}\left(r_{0,j} - jk_{N_k}\right)\right] \\ \mathrm{Im}\left[Q_{ij}(r_{0,j} - jk_1)\right] \\ \vdots \\ \mathrm{Im}\left[Q_{ij}\left(r_{0,j} - jk_{N_k}\right)\right] \end{bmatrix} \tag{A8}$$

$$A = \begin{bmatrix} 1 & 0 & \mathrm{Re}\left[(jk_1)^2\right] & 0 \\ & & \vdots & \\ 1 & 0 & \mathrm{Re}\left[\left(jk_{N_k}\right)^2\right] & 0 \\ 0 & \mathrm{Im}\left[jk_1\right] & 0 & \mathrm{Im}\left[(jk_1)^3\right] \\ & & \vdots & \\ 0 & \mathrm{Im}\left[jk_{N_k}\right] & 0 & \mathrm{Im}\left[\left(jk_{N_k}\right)^3\right] \end{bmatrix} \tag{A9}$$

$$X_j = \begin{bmatrix} P_{0,ij} \\ P_{1,ij} \\ P_{2,ij} \\ P_{3,ij} \end{bmatrix} \tag{A10}$$

where $P_{m,ij}$ is the ith, jth element of $[P_m]$. If the constraint that is

$$||X_j||^2 = \text{Minimum} \tag{A11}$$

imposed so that the requirement of the rank of the matrix A in equation (A9) does not have to equal the dimension of X_j, the unique solution to equation (A7) exists as (ref. 9)

$$X_j = A^{\dagger} B_j \qquad \text{(A12)}$$

where $A^{\dagger}$ is the pseudoinverse of A in equation (A9).

The minimization of equation (A6) is performed by a numerically constrained gradient procedure described in reference 10 for the variables of equation (A2). The constraint that must be enforced is that the resulting approximation function be stable. It is necessary and sufficient for the function to be stable if the roots of the functions described by equation (A2) have negative real parts. For the example presented in the appendix this requires that

$$r_{0,j} < 0 \qquad \text{(A13)}$$

At each step in the parameter search for the minimum fo equation (A7), equation (A12) is used to solve for the variables in equation (A10). The procedure for higher order approximations is similar.

REFERENCES

1. Weiss, Steven J.; Tseng, Kadin; and Morino, Luigi: State-Space Formulations for Flutter Analysis. AIAA Paper 77-117, Jan. 1977.

2. Edwards, John William: Unsteady Aerodynamic Modeling and Active Aeroelastic Control. SUDAAR 504 (NASA Grant NGL-05-020-007), Stanford Univ., Feb. 1977. (Available as NASA CR-148019.)

3. Sevart, Francis D.: Development of Active Flutter Suppression Wind Tunnel Testing Technology. AFFDL-TR-74-126, U.S. Air Force, Jan. 1975.

4. Roger, Kenneth L.: Airplane Math Modeling Methods for Active Control Design. Structural Aspects of Active Controls, AGARD-CP-228, Aug. 1977, pp. 4-1 - 4-11.

5. Abel, Irving: An Analytical Technique for Predicting the Characteristics of a Flexible Wing Equipped With an Active Flutter-Suppression System and Comparison With Wind-Tunnel Data. NASA TP-1367, 1979.

6. Lyons, M. G.; Vepa, R.; McIntosh, S. C., Jr.; and DeBra, D. B.: Control Law Synthesis and Sensor Design for Active Flutter Suppression. AIAA Paper No. 73-832, Aug. 1973.

7. Vepa, Ranjan: On the Use of Pade Approximants To Represent Unsteady Aerodynamic Loads for Arbitrarily Small Motion of Wings. AIAA Paper No. 76-17, Jan. 1976.

8. Albano, Edward; and Rodden, William P.: A Doublet-Lattice Method for Calculating Lift Distributions on Oscillating Surfaces in Subsonic Flows. AIAA J., vol. 7, no. 2, Feb. 1969, pp. 279-285; Errata, vol. 7, no. 11, Nov. 1969, p. 2192.

9. Wilkinson, J. H.; and Reinsch, C.: Handbook for Automatic Computation. Volume II - Linear Algebra. Springer-Verlag, 1971.

10. Vanderplaats, Garret N.: CONMIN - A Fortran Program for Constrained Function Minimization. User's Manual. NASA TM X-62,282, 1973.

TABLE I.- FREQUENCY AND GENERALIZED MASS

Mode	Natural frequency, Hz	Generalized mass, kg
1	5.233	3.678
2	19.129	7.769
3	20.906	7.044
4	25.769	2.970
5	46.110	4.714
6	61.234	4.758
7	79.682	5.156
8	86.030	11.297
9	98.087	7.558
10	118.150	5.501

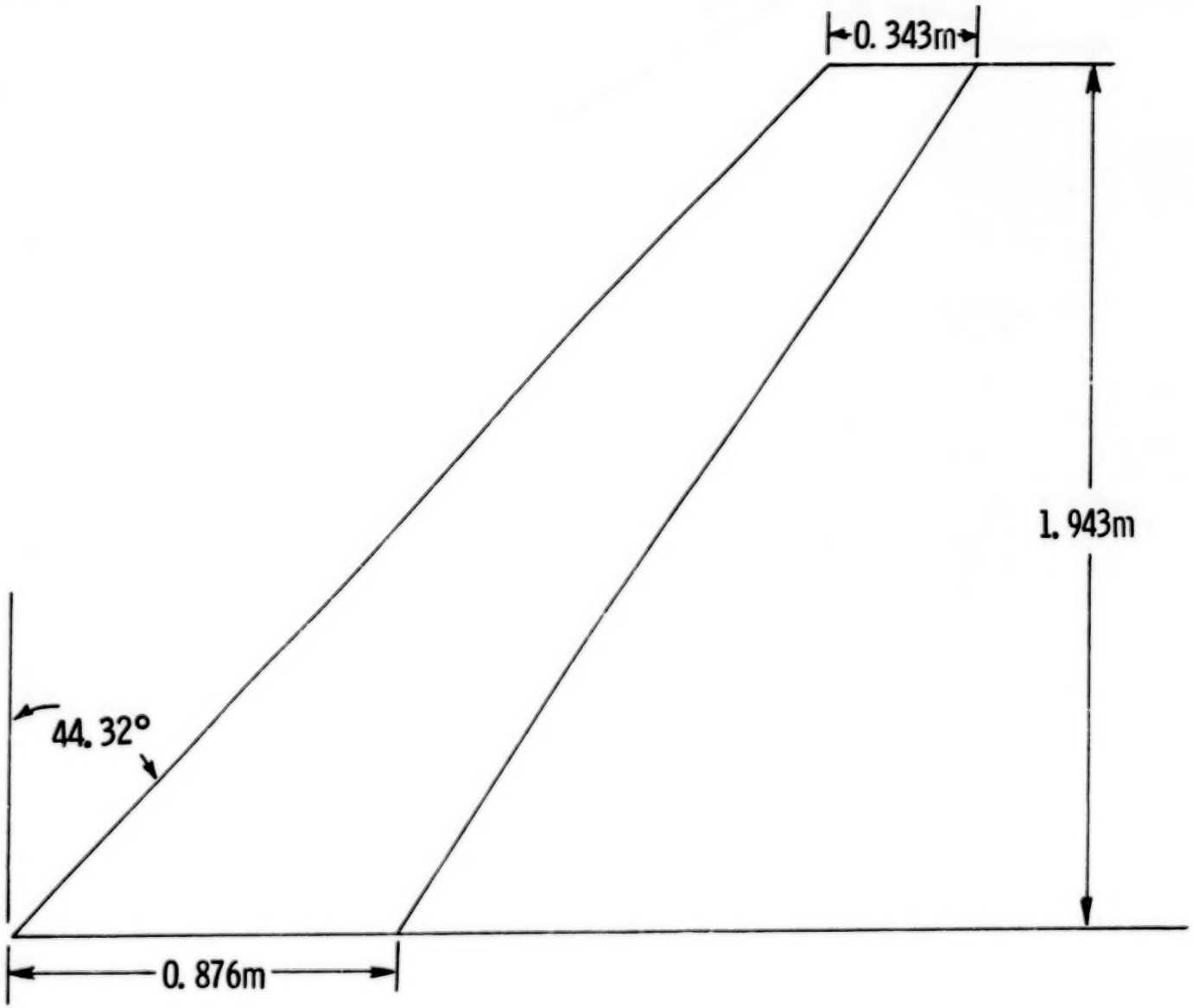


Figure 1.- Model planform.

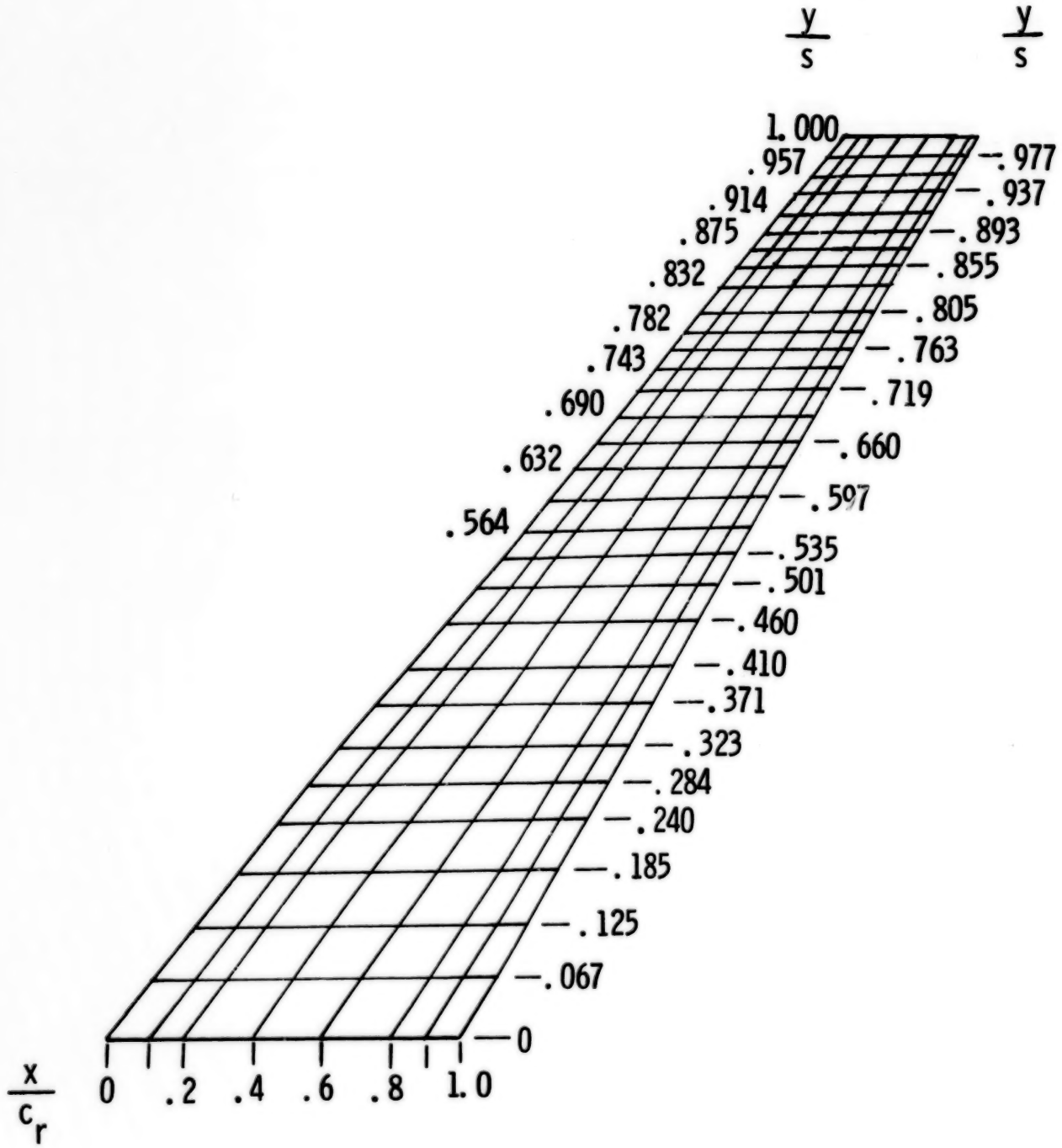
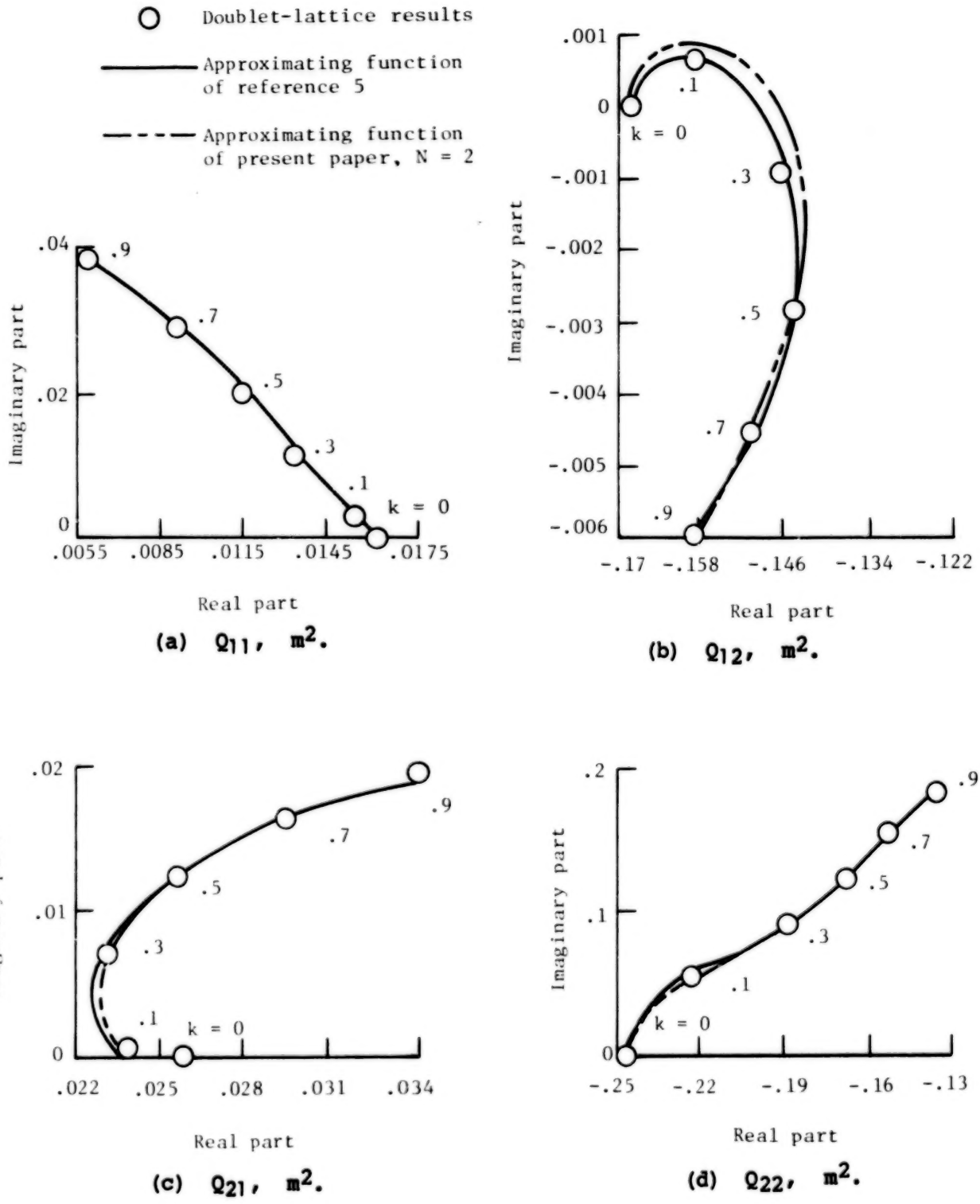


Figure 2.- Paneling scheme for doublet-lattice aerodynamics.



(a) Q_{11}, m^2.

(b) Q_{12}, m^2.

(c) Q_{21}, m^2.

(d) Q_{22}, m^2.

Figure 3.- Approximation for Q_{11}, Q_{12}, Q_{21}, and Q_{22} at M = 0.90 as a function of reduced frequency (reduced frequency at flutter equals 0.12).

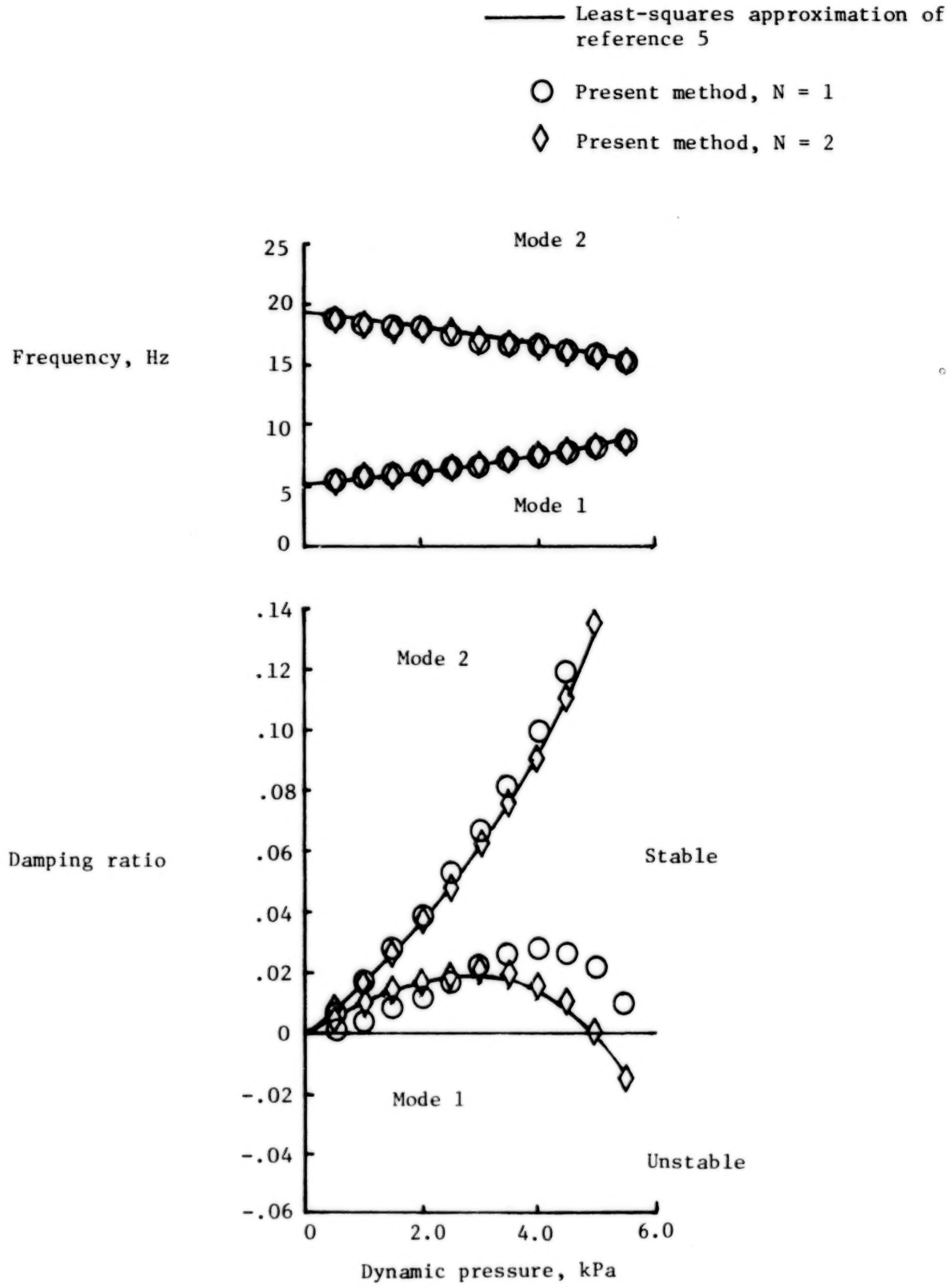


Figure 4.- Change in frequency and damping ratio with respect to change in dynamic pressure.

1. Report No. NASA TP-1738	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle AN ANALYTICAL TECHNIQUE FOR APPROXIMATING UNSTEADY AERODYNAMICS IN THE TIME DOMAIN		5. Report Date November 1980 6. Performing Organization Code
7. Author(s) H. J. Dunn		8. Performing Organization Report No. L-13255
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665		10. Work Unit No. 505-34-33-02 11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546		13. Type of Report and Period Covered Technical Paper 14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

An analytical technique is presented for approximating unsteady aerodynamic forces in the time domain. The order of elements of a matrix Padé approximation is postulated, and the resulting polynomial coefficients are determined through a combination of least-squares estimates for the numerator coefficients and a constrained gradient search for the denominator coefficients which insures stable approximating functions. The number of differential equations required to represent the aerodynamic forces to a given accuracy tends to be smaller than that employed in certain existing techniques where the denominator coefficients are chosen a priori. Results are shown for an aeroelastic, cantilevered, semispan wing which indicate a good fit to the aerodynamic forces for oscillatory motion can be achieved with a matrix Padé approximation having fourth-order numerator and second-order denominator polynomials.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Aeroelasticity Modeling Time domain	Unclassified - Unlimited Subject Category 39

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	28	A03

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1980

END

March 6, 1981